EMBEDDINGS OF SURFACES INTO 3-SPACE AND QUADRUPLE POINTS OF REGULAR HOMOTOPIES

TAHL NOWIK

Abstract. Let F be a closed orientable surface. We give an explicit formula for the number mod 2 of quadruple points occurring in any generic regular homotopy between any two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$. The formula is in terms of homological data extracted from the two embeddings.

1. Introduction

For F a closed surface and $i, i' : F \to \mathbb{R}^3$ two regularly homotopic generic immersions, we are interested in the number mod 2 of quadruple points occurring in generic regular homotopies between i and i'. It has been shown in [N1] that this number is the same for all such regular homotopies, and so it is a function of i and i' which we denote $Q(i, i') \in \mathbb{Z}/2$. For F orientable and $e, e' : F \to \mathbb{R}^3$ two regularly homotopic *embeddings*, we give an explicit formula for $Q(e, e')$ which depends on the following data: If $e : F \to \mathbb{R}^3$ is an embedding then $e(F)$ splits $\mathbb{R}^3$ into two pieces, one compact which will be denoted $M^0(e)$ and the other non-compact which will be denoted $M^1(e)$. By restriction of range e induces maps $e^k : F \to M^k(e)$ $(k = 0, 1)$ and let $A^k(e) \subseteq H_1(F, \mathbb{Z}/2)$ be the kernel of the map induced by e^k on $H_1(\cdot, \mathbb{Z}/2)$. Let $o(e)$ be the orientation on F which is induced from $M^0(e)$ to $\partial M^0(e) = e(F)$ and then via e to F. Our formula for $Q(e, e')$ will be in terms of the two triplets $A^0(e), A^1(e), o(e)$ and $A^0(e'), A^1(e'), o(e')$. Our formula will be also easily extended to finite unions of closed orientable surfaces.

For two special cases a formula for $Q(e, e')$ (for e, e' embeddings) has already been known: The case where F is a sphere has appeared in [MB] and [N1], and the case where F is a torus has appeared in [N1]. The starting point for our work will be [N2] where an explicit formula has been given for $Q(i, i \circ h)$, where $i : F \to \mathbb{R}^3$ is any generic immersion and $h : F \to F$ is any diffeomorphism such that i and $i \circ h$ are regularly homotopic.

Date: November 9, 1999.

2. Totally Singular Decompositions

Let V be a finite dimensional vector space over $\mathbb{Z}/2$. A function $g : V \to \mathbb{Z}/2$ is called a *quadratic form* if g satisfies: $g(x+y) = g(x)+g(y)+B(x,y)$ for all $x, y \in V$, where $B(x, y)$ is a bilinear form. The following properties follow: (a) $g(0) = 0$. (b) $B(x, x) = 0$ for all $x \in V$. (c) $B(x, y) = B(y, x)$ for all $x, y \in V$. g is called *non-degenerate* if B is non-degenerate, i.e. for any $0 \neq x \in V$ there is $y \in V$ with $B(x, y) \neq 0$. For an exposition of quadratic forms see [C].

In what follows we always assume that our vector space V is equipped with a non-degenerate quadratic form g. It then follows that $\dim V$ is even. A subspace $A \subseteq V$ such that $g|_A \equiv 0$ is called a *totally singular* subspace. A pair (A, B) of subspaces of V will be called a *totally singular decomposition* (abbreviated TSD) of V if $V = A \oplus B$ and both A and B are totally singular. It then follows that $\dim A = \dim B$. (We remark that TSDs do not always exist. They will however always exist for the quadratic forms which will arise in our geometric considerations, as seen in Lemma 3.3 below.) A linear map $T : V \to V$ is called *orthogonal* if $g(T(x)) = g(x)$ for all $x \in V$. It then follows that $B(T(x), T(y)) = B(x, y)$ for all $x, y \in V$ and that T is invertible. The group of all orthogonal maps of V with respect to g will be denoted $O(V, g)$.

The proof of the following lemma appears in [C]:

Lemma 2.1. *Let* $\dim V = 2n$*.*

1. *If $A \subseteq V$ is a totally singular subspace of dimension n then there exists a $B \subseteq V$ such that (A, B) is a TSD of V.*
2. *If (A, B) is a TSD of V and $a_1, \dots, a_n$ is a given basis for A then there is a basis $b_1, \dots, b_n$ for B such that $B(a_i, b_j) = \delta_{ij}$.*

Definition 2.2. If (A, B) is a TSD of V then a basis $a_1, \dots, a_n, b_1, \dots, b_n$ of V will be called (A, B)*-good* if $a_i \in A$, $b_i \in B$ and $B(a_i, b_j) = \delta_{ij}$.

The following two lemmas follow directly from the definition of quadratic form:

Lemma 2.3. *Let (A, B) be a TSD of V and $a_1, \dots, a_n, b_1, \dots, b_n$ an (A, B)-good basis for V. If $v = \sum x_i a_i + \sum y_i b_i$ and $v' = \sum x'_i a_i + \sum y'_i b_i$ then $g(v) = \sum x_i y_i$ and $B(v, v') = \sum x_i y'_i + \sum y_i x'_i$.*

Lemma 2.4. *Let (A, B) and (A', B') be two TSDs of V. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be an (A, B)-good basis for V and $a'_1, \dots, a'_n, b'_1, \dots, b'_n$ an (A', B')-good basis for V. If $T : V \to V$ is the linear map defined by $a_i \mapsto a'_i$, $b_i \mapsto b'_i$ then $T \in O(V, g)$.*

For $T \in O(V, g)$ we define $\psi(T) \in \mathbb{Z}/2$ by:

$$\psi(T) \;=\; \operatorname{rank}(T - Id) \mod 2.$$

It has been shown in [N2] that $\psi : O(V, g) \to \mathbb{Z}/2$ is a (non-trivial) homomorphism.

Lemma 2.5. *If (A, B) is a TSD of V and $T \in O(V, g)$ satisfies $T(A) = A$ and $T(B) = B$ then $\psi(T) = 0$.*

Proof. By Lemma 2.1 there exists an (A, B)-good basis $a_1, \dots, a_n, b_1, \dots, b_n$ for V. Using Lemma 2.3 it is easy to verify that the matrix of T with respect to such a basis has the form: $\begin{pmatrix} S^t & 0 \\ 0 & S^{-1} \end{pmatrix}$ where $S \in GL_n(\mathbb{Z}/2)$. It follows that $\psi(T) = 0$. □

Given two TSDs (A, B), (A', B') of V then by Lemmas 2.1 and 2.4 there exists a $T \in O(V, g)$ such that $T(A) = A'$ and $T(B) = B'$. It follows from Lemma 2.5 that if T_1, T_2 are two such Ts then $\psi(T_1) = \psi(T_2)$. And so the following is well defined:

Definition 2.6. For a pair (A, B), (A', B') of TSDs of V let $\widehat{\psi}(A, B; A', B') \in \mathbb{Z}/2$ be defined by $\widehat{\psi}(A, B; A', B') = \psi(T)$ for some (thus all) $T \in O(V, g)$ with $T(A) = A'$ and $T(B) = B'$.

Definition 2.7. For two TSDs (A, B), (A', B') of V, we will write $(A, B) \sim (A', B')$ if $\widehat{\psi}(A, B; A', B') = 0$.

Since ψ is a homomorphism, $\widehat{\psi}(A, B; A'', B'') = \widehat{\psi}(A, B; A', B') + \widehat{\psi}(A', B'; A'', B'')$ for any three TSDs $(A, B), (A', B'), (A'', B'')$. It follows that $\sim$ is an equivalence relation with precisely two equivalence classes and that $\widehat{\psi}(A, B; A'', B'') = \widehat{\psi}(A', B'; A'', B'')$ whenever $(A, B) \sim (A', B')$.

Lemma 2.8. *Let $\dim V = 2n$ and let $A \subseteq V$ be a totally singular subspace of dimension n. If $T \in O(V, g)$ satisfies $T(x) = x$ for every $x \in A$ then $\psi(T) = 0$.*

Proof. By Lemma 2.1 there is a $B \subseteq V$ such that (A, B) is a TSD of V and an (A, B)-good basis $a_1, \dots, a_n, b_1, \dots, b_n$ for V. Using Lemma 2.3 it is easy to verify that the matrix of T

with respect to such a basis has the form: $\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$ where I is the $n \times n$ identity matrix and $S \in M_n(\mathbb{Z}/2)$ is an alternating matrix, i.e. if $S = \{s_{ij}\}$ then $s_{ii} = 0$ and $s_{ij} = s_{ji}$. Since alternating matrices have even rank, it follows that $\psi(T) = 0$. □

Corollary 2.9. *Let (A, B) and (A', B') be two TSDs of V. If $A = A'$ or $B = B'$ then $(A, B) \sim (A', B')$.*

Proof. Say $A = A'$. By Lemmas 2.1 and 2.4 there exists a $T \in O(V, g)$ with $T(x) = x$ for all $x \in A = A'$ and $T(B) = B'$. The conclusion follows from Lemma 2.8. □

Let $V_0, V_1 \subseteq V$ be two subspaces of V. We will write $V_0 \perp V_1$ if $B(x, y) = 0$ for every $x \in V_0$, $y \in V_1$. The following is clear:

Lemma 2.10. *Let $V_0, V_1 \subseteq V$ satisfy $V = V_0 \oplus V_1$ and $V_0 \perp V_1$.*

1. *If for $l = 0, 1$, (A_l, B_l) is a TSD of V_l (with respect to $g|_{V_l}$ which is indeed non-degenerate) then $(A_0 + A_1, B_0 + B_1)$ is a TSD of V.*
2. *If (A'_l, B'_l) is another TSD of V_l and $(A_l, B_l) \sim (A'_l, B'_l)$ ($l = 0, 1$) then $(A_0 + A_1, B_0 + B_1) \sim (A'_0 + A'_1, B'_0 + B'_1)$.*

3. Statement of Main Result

A *surface* is by definition assumed connected. A finite union of surfaces will be called a *system of surfaces*. Let S be a system of closed surfaces and $H_t : S \to \mathbb{R}^3$ a generic regular homotopy. We denote by $q(H_t) \in \mathbb{Z}/2$ the number mod 2 of quadruple points occurring in H_t. The following has been shown in [N1]:

Theorem 3.1. *Let S be a system of closed surfaces (not necessarily orientable.) If $H_t, G_t : S \to \mathbb{R}^3$ are two generic regular homotopies between the same two generic immersions, then $q(H_t) = q(G_t)$.*

Definition 3.2. Let S be a system of closed surfaces and $i, i' : S \to \mathbb{R}^3$ two regularly homotopic generic immersions. We define $Q(i, i') \in \mathbb{Z}/2$ by $Q(i, i') = q(H_t)$, where H_t is any generic regular homotopy between i and i'. This is well defined by Theorem 3.1.

Let F from now on denote a closed orientable surface. A simple closed curve in F will be called a *circle*. If c is a circle in F, the homology class of c in $H_1(F, \mathbb{Z}/2)$ will be denoted

by $[c]$. Any immersion $i : F \to \mathbb{R}^3$ induces a quadratic form $g^i : H_1(F, \mathbb{Z}/2) \to \mathbb{Z}/2$ whose associated bilinear form $B(x, y)$ is the algebraic intersection form $x \cdot y$ of $H_1(F, \mathbb{Z}/2)$, as follows: For $x \in H_1(F, \mathbb{Z}/2)$ let $A \subseteq F$ be an annulus bounded by circles c, c' with $[c] = x$, let $j : A \to \mathbb{R}^3$ be an embedding which is regularly homotopic to $i|_A$ and define $g^i(x)$ to be the $\mathbb{Z}/2$ linking number between $j(c)$ and $j(c')$ in $\mathbb{R}^3$. One needs to verify that $g^i(x)$ is independent of the choices being made and that $g^i(x + y) = g^i(x) + g^i(y) + x \cdot y$. This has been done in [P]. Also, $i, i' : F \to \mathbb{R}^3$ are regularly homotopic iff $g^i = g^{i'}$.

If $e : F \to \mathbb{R}^3$ is an embedding then $e(F)$ splits $\mathbb{R}^3$ into two pieces one compact and one non-compact. We denote the compact piece by $M^0(e)$ and the non-compact piece by $M^1(e)$. By restriction of range, e induces maps $e^k : F \to M^k(e)$, $k = 0, 1$. Let $e^k_* : H_1(F, \mathbb{Z}/2) \to H_1(M^k(e), \mathbb{Z}/2)$ be the maps induced on homology and finally let $A^k(e) = \ker e^k_*$, $k = 0, 1$.

Lemma 3.3. *Let $e : F \to \mathbb{R}^3$ be an embedding, then $(A^0(e), A^1(e))$ is a TSD of $H_1(F, \mathbb{Z}/2)$ with respect to the quadratic form g^e.*

Proof. We first show that each $A^k(e)$ is totally singular: For $x \in A^k(e)$ let A, c, c' be as in the definition of $g^e(x)$ and simply take $j = e|_A$. Since $e^k_*(x) = 0$, $e(c)$ bounds a properly embedded (perhaps non-orientable) surface S in $M^k(e)$. Since $e(c')$ is disjoint from S, the $\mathbb{Z}/2$ linking number between $e(c)$ and $e(c')$ in $\mathbb{R}^3$ is 0, and so $g^e(x) = 0$. Now, the fact that $H_1(F, \mathbb{Z}/2) = A^0(e) \oplus A^1(e)$ is a consequence of the $\mathbb{Z}/2$ Mayer-Vietoris sequence for $\mathbb{R}^3 = M^0(e) \cup M^1(e)$ where F is identified with $M^0(e) \cap M^1(e)$ via e. □

If $e, e' : F \to \mathbb{R}^3$ are two regularly homotopic embeddings then $g^e = g^{e'}$ so $(A^0(e), A^1(e))$ and $(A^0(e'), A^1(e'))$ are TSDs of $H_1(F, \mathbb{Z}/2)$ with respect to the same quadratic form and so $\widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e'))$ is defined. We spell out the actual computation involved in $\widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e'))$:

1. Find a basis $a_1, \dots, a_n, b_1, \dots, b_n$ for $H_1(F, \mathbb{Z}/2)$ such that $e^0_*(a_i) = 0$, $e^1_*(b_i) = 0$ and $a_i \cdot b_j = \delta_{ij}$.
2. Find a similar basis $a'_1, \dots, a'_n, b'_1, \dots, b'_n$ using e' in place of e.
3. Let m be the dimension of the subspace of $H_1(F, \mathbb{Z}/2)$ spanned by:

$$a'_1 - a_1 \ , \ \dots \ , \ a'_n - a_n \ , \ b'_1 - b_1 \ , \ \dots \ , \ b'_n - b_n.$$

4. $\widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) = m \bmod 2$, (an element in $\mathbb{Z}/2$.)

Definition 3.4. If $e : F \to \mathbb{R}^3$ is an embedding then we define $o(e)$ to be the orientation on F which is induced from $M^0(e)$ to $\partial M^0(e) = e(F)$ and then via e to F (and where the orientation on $M^0(e)$ is the restriction of the orientation of $\mathbb{R}^3$.) If $e, e' : F \to \mathbb{R}^3$ are two embeddings then we define $\widehat{\epsilon}(e, e') \in \mathbb{Z}/2$ to be 0 if $o(e) = o(e')$ and 1 if $o(e) \neq o(e')$.

Our purpose in this work is to show:

Theorem 3.5. *Let n be the genus of F. If $e, e' : F \to \mathbb{R}^3$ are two regularly homotopic embeddings then:*

$$Q(e, e') \ = \ \widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) \ + \ (n+1)\widehat{\epsilon}(e, e').$$

Our starting point is the following theorem which has been proved in [N2]:

Theorem 3.6. *For any generic immersion $i : F \to \mathbb{R}^3$ and any diffeomorphism $h : F \to F$ such that i and $i \circ h$ are regularly homotopic,*

$$Q(i, i \circ h) = \psi(h_*) + (n+1)\epsilon(h),$$

where h_* is the map induced by h on $H_1(F, \mathbb{Z}/2)$, n is the genus of F and $\epsilon(h) \in \mathbb{Z}/2$ is 0 or 1 according to whether h is orientation preserving or reversing, respectively.

4. Equivalent Embeddings and k-Extendible Regular Homotopies

Let $e : F \to \mathbb{R}^3$ be an embedding, let $P \subseteq \mathbb{R}^3$ be a plane and assume $e(F)$ intersects P transversally in a unique circle. Let $c = e^{-1}(P)$ then c is a separating circle in F. Let A be a regular neighborhood of c in F and let F_0, F_1 be the connected components of $F - intA$. (A lower index will always be related to the splitting of $\mathbb{R}^3$ via a plane, the assignment of 0 and 1 to the two sides being arbitrary. An upper index on the other hand is related to the splitting of $\mathbb{R}^3$ via the image of a closed surface, assigning 0 to the compact side and 1 to the non-compact side.) Let $\bar{F}_l$ $(l = 0, 1)$ be the closed surface obtained by gluing a disc D_l to F_l. Let $e_l : \bar{F}_l \to \mathbb{R}^3$ be the embedding such that $e_l|_{F_l} = e|_{F_l}$ and $e_l(D_l)$ is parallel to P. Let $i_{F_l F} : F_l \to F$ and $i_{F_l \bar{F}_l} : F_l \to \bar{F}_l$ denote the inclusion maps. The induced map $i_{F_l \bar{F}_l *} : H_1(F_l, \mathbb{Z}/2) \to H_1(\bar{F}_l, \mathbb{Z}/2)$ is an isomorphism and let $h_l : H_1(\bar{F}_l, \mathbb{Z}/2) \to H_1(F, \mathbb{Z}/2)$ be the map $h_l = i_{F_l F *} \circ (i_{F_l \bar{F}_l *})^{-1}$.

Lemma 4.1. *Under the above assumptions and definitions: $A^k(e) = h_0(A^k(e_0)) + h_1(A^k(e_1))$, $k = 0, 1$.*

Proof. This follows from the fact that the inclusions $F_0 \cup F_1 \to \bar{F}_0 \cup \bar{F}_1$, $F_0 \cup F_1 \to F$, $M^0(e_0) \cup M^0(e_1) \to M^0(e)$ and $M^1(e) \to \mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ all induce isomorphisms on $H_1(\cdot, \mathbb{Z}/2)$ and the splitting of each of the above spaces via P induces a direct sum decomposition. We only check that the inclusion $M^1(e) \to \mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ induces isomorphism on $H_1(\cdot, \mathbb{Z}/2)$. Indeed $\mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ is obtained from $M^1(e)$ by gluing a 2-handle along $e(A)$, and the inclusion of $e(A)$ in $M^1(e)$ is null-homotopic. □

Definition 4.2. Two embeddings $e, f : F \to \mathbb{R}^3$ will be called *equivalent* if:

1. There is a regular homotopy between e and f with no quadruple points.
2. $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$.
3. $o(e) = o(f)$

Definition 4.3. An embedding $e : F \to \mathbb{R}^3$ will be called *standard* if its image $e(F)$ is a surface in $\mathbb{R}^3$ as in Figure 1.

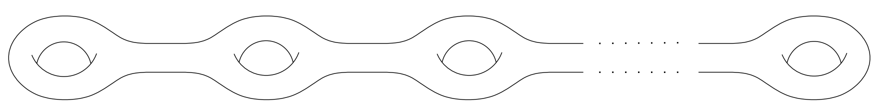

FIGURE 1. Image of a standard embedding.

In Proposition 4.8 below we will show that any embedding $e : F \to \mathbb{R}^3$ is equivalent to a standard embedding. The following lemma will be used in the induction step:

Lemma 4.4. *Let $e : F \to \mathbb{R}^3$ be an embedding. Assume $e(F)$ intersects a plane $P \subseteq \mathbb{R}^3$ transversally in one circle and let $c, A, F_l, \bar{F}_l, D_l, e_l$ be as above. If $e_l : \bar{F}_l \to \mathbb{R}^3$ $(l = 0, 1)$ are both equivalent to standard embeddings, then e is equivalent to a standard embedding.*

Proof. Changing e by isotopy, we may assume $e(A)$ is a very thin tube. $e_l : \bar{F}_l \to \mathbb{R}^3$ is equivalent to a standard embedding f_l via a regular homotopy $(H_l)_t : \bar{F}_l \to \mathbb{R}^3$ having no quadruple points. We may further assume that each $(H_l)_t$ moves $\bar{F}_l$ only within the corresponding half-space defined by P, that each $f_l(D_l)$ is situated at the point of $f_l(\bar{F}_l)$ which is closest to P and that these two points are opposite each other with respect to P. We now perform both $(H_l)_t$, letting the thin tube A be carried along. If we make sure the thin tube A does not pass triple points occurring in F_1 and F_2 then the regular

homotopy H_t induced on F in this way will also have no quadruple points. Since $e(A)$ has approached $e_l(\bar{F}_l)$ from $M^1(e_l)$ and since $o_{e_l} = o_{f_l}$, we also have at the end of H_t that A approaches $f_l(\bar{F}_l)$ from $M^1(f_l)$. And so we may continue moving the tube A until it is all situated in the region between $f_0(\bar{F}_0)$ and $f_1(\bar{F}_1)$, then canceling all knotting by having the thin tube pass itself (this involves only double lines) until A is embedded as a straight tube connecting $f_0(\bar{F}_0)$ to $f_1(\bar{F}_1)$ and so the final map $f : F \to \mathbb{R}^3$ thus obtained is indeed a standard embedding. By assumption $(A^0(e_l), A^1(e_l)) \sim (A^0(f_l), A^1(f_l))$, $l = 0, 1$ which implies that $(h_l(A^0(e_l)), h_l(A^1(e_l))) \sim (h_l(A^0(f_l)), h_l(A^1(f_l)))$, $l = 0, 1$ as TSDs of $V_l = h_l(H_1(\bar{F}_l, \mathbb{Z}/2)) \subseteq H_1(F, \mathbb{Z}/2)$. (Note that h_l preserves the corresponding quadratic forms.) But $H_1(F, \mathbb{Z}/2) = V_0 \oplus V_1$ and $V_0 \perp V_1$ and so by Lemma 2.10 and Lemma 4.1 $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$. Finally, from $o_{e_l} = o_{f_l}$ it follows that $o(e) = o(f)$. $\square$

Definition 4.5. Let $e, f : F \to \mathbb{R}^3$ be two embeddings. A regular homotopy $H_t : F \to \mathbb{R}^3$ $(a \leq t \leq b)$ with $H_a = e$, $H_b = f$ will be called *k-extendible* (where k is either 0 or 1) if there exists a regular homotopy $G_t : M^k(e) \to \mathbb{R}^3$ $(a \leq t \leq b)$ satisfying:

1. G_a is the inclusion map of $M^k(e)$ in $\mathbb{R}^3$.
2. $H_t = G_t \circ e^k$. (Recall that $e^k : F \to M^k(e)$ is simply e with range restricted to $M^k(e)$.)
3. G_b is an embedding with $G_b(M^k(e)) = M^k(f)$.

Lemma 4.6. *If for a given k there is a k-extendible regular homotopy between the embeddings e and f then $A^k(e) = A^k(f)$.*

Proof. $f = H_b = G_b \circ e^k$ and so $f^k = G_b^k \circ e^k$ where $G_b^k : M^k(e) \to M^k(f)$ is the map G_b with range restricted to $M^k(f)$. Since G_b^k is a diffeomorphism it follows that $\ker f_*^k = \ker e_*^k$. $\square$

Corollary 4.7. *If there is a k-extendible regular homotopy between the embeddings e and f for either $k = 0$ or $k = 1$ then:*

1. $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$.
2. $o(e) = o(f)$.

Proof. 1 follows from Lemma 4.6 and Corollary 2.9. Since G_a is the inclusion and G_t is a regular homotopy it follows that G_b is orientation preserving. This implies 2. $\square$

Proposition 4.8. *Every embedding $e : F \to \mathbb{R}^3$ is equivalent to a standard embedding.*

Proof. The proof is by induction on the genus of F. If $F = S^2$ then any e is isotopic to a standard embedding and isotopic embeddings are equivalent. So assume F is of positive genus and so there is a compressing disc D for $e(F)$ in $\mathbb{R}^3$ (i.e. $D \cap e(F) = \partial D$ and ∂D does not bound a disc in $e(F)$.) Let $c = e^{-1}(\partial D) \subseteq F$ and let A be a regular neighborhood of c in F. Isotoping A along D as before we may assume A is embedded as a thin tube. There are four cases to be considered according to whether D is contained in $M^0(e)$ or $M^1(e)$ and whether ∂D separates or does not separate $e(F)$.

Case 1: $D \subseteq M^0(e)$ and ∂D separates $e(F)$. It then follows that D separates $M^0(e)$. If F_0, F_1 denote the two components of $F - intA$ and $e_l : \bar{F}_l \to \mathbb{R}^3$ are defined as before then it follows from the assumptions of this case that $M^0(e_0)$ and $M^0(e_1)$ are disjoint and the tube $e(A)$ approaches each $e_l(\bar{F}_l)$ from its non-compact side, i.e. from $M^1(e_l)$. Move each foot of the tube $e(A)$ (see Figure 2) along the corresponding surface $e_l(\bar{F}_l)$ until they are each situated at the point p_l of $e_l(\bar{F}_l)$ having maximal z-coordinate. In particular it follows that now $e(A)$ approaches each $e_l(\bar{F}_l)$ from above. We now uniformly shrink each $e(F_l)$ towards the point p_l until it is contained in a tiny ball B_l attached from below to the corresponding foot of $e(A)$, arriving at a new embedding $e' : F \to \mathbb{R}^3$. This regular homotopy is clearly 0-extendible, and since no self intersection may occur within each of F_0, F_1 and A, this regular homotopy has no quadruple points. And so by Corollary 4.7 e' is equivalent to e. We now continue by isotopy, deforming the thin tube $e'(A)$ until it is a straight tube, and rigidly carrying B_0 and B_1 along. We finally arrive at an embedding e'' for which there is a plane P intersecting $e''(F)$ as in Lemma 4.4 with our F_0 and F_1 on the two sides of P. Since the genus of both $\bar{F}_0$ and $\bar{F}_1$ is smaller than that of F then by our induction hypothesis and Lemma 4.4, e'' is equivalent to a standard embedding.

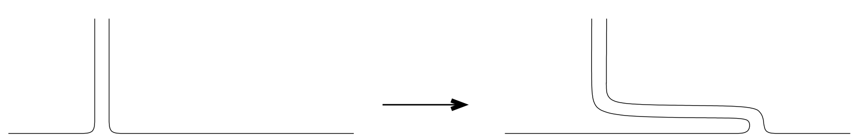

FIGURE 2. Moving the foot of a tube.

Case 2: $D \subseteq M^1(e)$ and ∂D separates $e(F)$. This time either $M^0(e_0) \subseteq M^0(e_1)$ or $M^0(e_1) \subseteq M^0(e_0)$ and assume the former holds. In this case $e(A)$ approaches only $e_0(\bar{F}_0)$ from its non-compact side and so we push the tube and perform the uniform shrinking as above only with F_0. This is a 1-extendible regular homotopy since we are shrinking $M^0(e_0)$

which is part of $M^1(e)$. Now, if B is the tiny ball into which we have shrunken $e(F_0)$ then ∂B supplies separating compressing discs on both sides of $e(F)$ and so we are done by Case 1.

Case 3: $D \subseteq M^0(e)$ and ∂D does not separate $e(F)$. If $F' = F - intA$ and $e' : \bar{F}' \to \mathbb{R}^3$ is induced as above (where $\bar{F}'$ is the surface obtained from F' by gluing two discs to it) then both feet of the tube $e(A)$ approach $e'(\bar{F}')$ from its non-compact side. Push the feet of $e(A)$ until they are both situated near the same point p in $e'(\bar{F}')$ having maximal z coordinate. Let P be a horizontal plane passing slightly below p (so that in a neighborhood of p it intersects F in only one circle.) We may pull the tube $e(A)$ until it is all above P. We then let it pass through itself until it is unknotted. This is a 0-extendible regular homotopy with no quadruple points, at the end of which we have an embedding intersecting P as in Lemma 4.4 with an embedding of a torus above the plane P, this embedding being already standard and an embedding of a subsurface F'' of F below the plane P, F'' being of smaller genus than that of F. Again we are done by induction and Lemma 4.4.

Case 4: $D \subseteq M^1(e)$ and ∂D does not separate $e(F)$. We may proceed as in Case 3 (this time via a 1-extendible regular homotopy) to obtain a standard embedding of a torus connected with a tube to $e'(\bar{F}')$ but this time the torus is contained in $M^0(e')$ and the tube connects to $e'(\bar{F}')$ from its compact side. But once we have such an embedding then the little standardly embedded torus has non-separating compressing discs on both sides and so we are done by Case 3.

□

Lemma 4.9. *If $e : F \to \mathbb{R}^3$ is an embedding and $h : F \to F$ is a diffeomorphism such that e and $e \circ h$ are regularly homotopic, then $\widehat{\psi}(A^0(e), A^1(e); A^0(e \circ h), A^1(e \circ h)) = \psi(h_*)$ and $\widehat{\epsilon}(e, e \circ h) = \epsilon(h)$. (Recall that h_* is the map induced by h on $H_1(F, \mathbb{Z}/2)$ and $\epsilon(h) \in \mathbb{Z}/2$ is 0 or 1 according to whether h is orientation preserving or reversing.)*

Proof. $x \in \ker(e \circ h)^k_*$ iff $h_*(x) \in \ker e^k_*$ and so $A^k(e \circ h) = h_*^{-1}(A^k(e))$, $k = 0, 1$. By definition then $\widehat{\psi}(A^0(e), A^1(e); A^0(e \circ h), A^1(e \circ h)) = \psi(h_*^{-1}) = \psi(h_*)$. (Note that if e and $e \circ h$ are regularly homotopic then indeed $h_*^{-1} \in O(H_1(F, \mathbb{Z}/2), g^e)$.) $\widehat{\epsilon}(e, e \circ h) = \epsilon(h)$ is clear. □

We are now ready to prove Theorem 3.5. For two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$ let $\widehat{\Psi}(e, e') = \widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) + (n + 1)\widehat{\epsilon}(e, e')$. We need to show $Q(e, e') = \widehat{\Psi}(e, e')$. If $e'' : F \to \mathbb{R}^3$ is also in the same regular homotopy class then $Q(e, e'') =$

$Q(e,e') + Q(e',e'')$ and $\widehat{\Psi}(e,e'') = \widehat{\Psi}(e,e') + \widehat{\Psi}(e',e'')$. And so if e' is equivalent to e'' and $Q(e,e'') = \widehat{\Psi}(e,e'')$ then also $Q(e,e') = \widehat{\Psi}(e,e')$. And so we may replace e with an equivalent standard embedding f (Proposition 4.8) and similarly replace e' with an equivalent standard embedding f'. Now f and f' have isotopic images and so after isotopy we may assume $f(F) = f'(F)$ and so $f' = f \circ h$ for some diffeomorphism $h : F \to F$. By Lemma 4.9 and Theorem 3.6 the proof of Theorem 3.5 is complete.

We conclude with a remark on systems of surfaces. If $S = F_1 \cup \cdots \cup F_r$ is a system of closed orientable surfaces, and $e : S \to \mathbb{R}^3$ is an embedding, then we can rigidly move $e(F_i)$ one by one, until they are all contained in large disjoint balls. When it is the turn of F_i to be rigidly moved, then the union of all other components is embedded and so only double lines occur. If $e' : S \to \mathbb{R}^3$ is another embedding then we can similarly move $e'(F_i)$ into the corresponding balls. It follows that $Q(e,e') = \sum_{i=1}^r Q(e|_{F_i}, e'|_{F_i})$ and so we obtain a formula for systems of surfaces, namely: $Q(e,e') = \sum_{i=1}^r \widehat{\Psi}(e|_{F_i}, e'|_{F_i})$.

References

[C] C.C. Chevalley: "The Algebraic Theory of Spinors." Columbia University Press 1954. Also reprinted in: C. Chevalley: "The Algebraic Theory of Spinors and Clifford Algebras, Collected Works, Vol. 2" Springer-Verlag 1997.

[MB] N. Max, T. Banchoff: "Every Sphere Eversion Has a Quadruple Point." *Contributions to Analysis and Geometry*, John Hopkins University Press, (1981), 191-209.

[N1] T. Nowik: "Quadruple Points of Regular Homotopies of Surfaces in 3-Manifolds." *Topology* - to appear. (may be viewed at: http://www.math.columbia.edu/˜tahl/publications.html)

[N2] T. Nowik: "Subgroups of the Mapping Class Group and Quadruple Points of Regular Homotopies." Preprint. (may be viewed at: http://www.math.columbia.edu/˜tahl/publications.html)

[P] U. Pinkall: "Regular homotopy classes of immersed surfaces." *Topology* 24 (1985) No.4, 421–434.

Department of Mathematics, Columbia University, New York, NY 10027, USA.

E-mail address: tahl@math.columbia.edu